thermal conductivity. For example, for a stess amplitude of 2.84 lb/in.$^2$, an increase of about 40% in the peak temperature was found at the second thermal resonance frequency. At the thermal resonant frequencies, very sharp increases in temperatures were found. Also, when the constant and variable thermal conductivity solutions were compared, (both solved by the weighing technique) no significant shift in the thermal resonant frequency was observed. This is evident in Fig. 3.

Thermal conductivity of a solid rocket propellant is generally a weak function of temperature. This is perhaps the reason for the constant thermal conductivity assumption in most research efforts studying the phenomenon. However, it is found that at high temperatures, this assumption introduces errors of large magnitudes. Convergence was found to be much slower when thermal conductivity was considered as a function of temperature than when it was fixed. The temperatures were generally higher than those for constant thermal conductivity, especially at the thermal resonant frequencies.

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# Analytical Dynamic Model Improvement Using Vibration Test Data

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## Introduction

THE structural dynamic model of a modern air vehicle becomes critical as the mission requirement becomes more stringent. Ground vibration tests are often used to improve the analytical dynamic model.[1-6]

The mode shapes and natural frequencies obtained from incomplete modal tests do not usually satisfy the dynamic equation and orthogonality requirements. The most common approach is first to modify the analytical mass matrix to satisfy the orthogonality condition based on the measured modal data. Then, the stiffness matrix is modified to fulfill the eigenvalue equation as a function of the measured mode shapes, natural frequencies, and the corrected mass matrix.[7-13]

An alternate approach is to correct the stiffness matrix using static test data. Then, the analytical mass matrix is modified to fulfill the eigenvalue equation based on the modal test data and the updated stiffness matrix.[14]

Both approaches have the merits of improving the analytical structural dynamic models; however, the interaction between mass and stiffness matrices are not taken into consideration in deriving the analytical equations. In this paper, both the analytical mass and stiffness matrices are modified simultaneously using the vibration test data.[15] The dynamic model improvement is obtained using the element correction method combined with the Lagrange multiplier technique.[16] The dynamic equation and the orthogonality constraints are enforced during the analytical derivation. The effects due to mass and stiffness interaction are clearly determined from the final equations. This method is a viable technique for improving an analytical model based on an incomplete set of test data.

## Theoretical Formulation

The mode shapes and natural frequencies obtained from ground vibration tests are often incomplete. The desired mass and stiffness matrices are required to be modified to fulfill the eigenvalue equation and the orthogonality constraints using the modal test data.

The measured modal matrix $\Phi$ $(n \times m)$ is rectangular, where $n \geq m$, and the natural frequencies matrix $\Omega^2$ $(m \times m)$ is diagonal. Both analytical mass $M_A$ $(n \times n)$ and stiffness $K_A$ $(n \times n)$ matrices are symmetric. The normalized mode shapes and the measured frequencies are assumed correct and have to satisfy the basic orthogonality requirement and eigenvalue equation, as shown in Eqs. (1) and (2).

$$\Phi^T M \Phi = I \tag{1}$$

$$K\Phi = M\Phi\Omega^2 \tag{2}$$

Where $M(n \times n)$ and $K(n \times n)$ are symmetric matrices and represent the corrected mass and stiffness matrices, respectively,

$$M = M^T \tag{3}$$

$$K = K^T \tag{4}$$

It is physically reasonable and mathematically convenient to correct the mass and stiffness matrices, which are subjected to the constraint Eqs. (1-4) by minimizing the weighted Euclidean norms, $\epsilon_1$ and $\epsilon_2$.

$$\epsilon_1 = ½\,||A^{-½}(K-K_A)A^{-½}|| = ½\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\sum_{q=1}^{n} a_{iq}^{-½}\sum_{p=1}^{n}\left(k_{qp}-k_{A_{qp}}\right)a_{pj}^{-½}\right]^2 \tag{5}$$

$$\epsilon_2 = ½\,||B^{-½}(M-M_A)B^{-½}|| = ½\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\sum_{q=1}^{n} b_{iq}^{-½}\sum_{p=1}^{n}\left(m_{qp}-m_{A_{qp}}\right)b_{pj}^{-½}\right]^2 \tag{6}$$

where matrices $A(n \times n)$ and $B(n \times n)$ are symmetric weighting functions for stiffness and mass matrices, respectively.

Received May 16, 1988; revision received Dec. 20, 1988. 

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Using Lagrange multipliers to include all the constraints of Eqs. (1-4), the Lagrange function ($\Psi$) is defined, as follows:

$$\Psi = \epsilon_1 + \epsilon_2 + 2\boldsymbol{I}\Delta(\Phi^T M\Phi - I)\boldsymbol{I} + 2\boldsymbol{I}\Lambda(K\Phi - M\Phi\Omega^2)\boldsymbol{I} + \boldsymbol{I}\beta_1(M - M^T)\boldsymbol{I} + \boldsymbol{I}\beta_2(K - K^T)\boldsymbol{I} \tag{7}$$

where

$$\boldsymbol{I}\Delta(\Phi^T M\Phi - I)\boldsymbol{I} = \sum_{t=1}^{m}\sum_{q=1}^{m}\Delta_{tq}\left(\sum_{i=1}^{n}\Phi_{ti}\sum_{j=1}^{n}m_{ij}\Phi_{jq} - \delta_{tq}\right) \tag{8}$$

$$\boldsymbol{I}\Lambda(K\Phi - M\Phi\Omega^2)\boldsymbol{I} = \sum_{p=1}^{n}\sum_{q=1}^{m}\lambda_{pq}\left(\sum_{i=1}^{n}k_{pi}\Phi_{iq} - \sum_{i=1}^{n}m_{pi}\sum_{s=1}^{m}\Phi_{is}\omega_{sq}^2\right) \tag{9}$$

$$\boldsymbol{I}\beta_1(M - M^T)\boldsymbol{I} = \sum_{i=1}^{n}\sum_{j=1}^{n}(\beta_1)_{ij}(m_{ij} - m_{ji}) \tag{10}$$

$$\boldsymbol{I}\beta_2(K - K^T)\boldsymbol{I} = \sum_{i=1}^{n}\sum_{j=1}^{n}(\beta_2)_{ij}(k_{ij} - k_{ji}) \tag{11}$$

The matrices $\Delta(m\times m)$, $\Lambda(n\times m)$, $\beta_1(n\times n)$, and $\beta_2(n\times n)$ are the Lagrange multipliers for Eqs. (1-4). Both matrices $\beta_1$ and $\beta_2$ are antisymmetric Lagrange multipliers and $\delta_{tq}$ is the Kronecker delta.

The partial differentiation of the Lagrange function $\Psi$ with respect to $m_{ij}$ and $k_{ij}$ is resulting in the $m_{ij}$ and $k_{ij}$ for minimum $\Psi$. In matrix form, the equations can be expressed as

$$\left[\frac{\partial\Psi}{\partial m_{ij}}\right] = B^{-1}(M - M_A)B^{-1} + 2\Phi\Delta\Phi^T + 2\beta_1 - 2\Lambda\Omega^2\Phi^T = 0 \tag{12}$$

$$\left[\frac{\partial\Psi}{\partial K_{ij}}\right] = A^{-1}(K - K_A)A^{-1} + 2\Lambda\Phi^T + 2\beta_2 = 0 \tag{13}$$

Eliminating the antisymmetric Lagrange multipliers $\beta_1$ and $\beta_2$ gives

$$M = M_A + B\Lambda\Omega^2\Phi^T B + B\Phi\Omega^2\Lambda^T B - B\Phi(\Delta + \Delta^T)\Phi^T B \tag{14}$$

$$K = K_A - A\Lambda\Phi^T A - A\Phi\Lambda^T A \tag{15}$$

Furthermore, substituting Eq. (14) into Eq. (1) to eliminate the Lagrange multiplier $(\Delta + \Delta^T)$ yields

$$M = M_A + B\Lambda Y + Y^T\Lambda^T B - M_0 - Z\Lambda Y - Y^T\Lambda^T Z \tag{16}$$

where

$$Y = \Omega^2\Phi^T B \tag{17}$$

$$M_0 = B\Phi Q^{-1}(\Phi^T M_A\Phi - I)Q^{-1}\Phi^T B \tag{18}$$

$$Z = B\Phi Q^{-1}\Phi^T B \tag{19}$$

$$Q = \Phi^T B\Phi \tag{20}$$

Since $M$ and $K$ are $n\times n$ symmetric matrices, Eqs. (15) and (16) can be separately represented as a column vector containing all the diagonal and right upper off-diagonal elements, as shown in Eqs. (21) and (22).

$$\{k_{ij}\} = \{k_{A_{ij}}\} - [U]\{\lambda_{pq}\} - [w]\{\lambda_{pq}\} \tag{21}$$

$$\{m_{ij}\} = \{m_{A_{ij}}\} + [G_1]\{\lambda_{pq}\} + [G_2]\{\lambda_{pq}\} - \{m_{0_{ij}}\} - [G_3]\{\lambda_{pq}\} - [G_4]\{\lambda_{pq}\} \tag{22}$$

where $\{m_{ij}\}$, $\{m_{A_{ij}}\}$, $\{k_{ij}\}$, $\{k_{A_{ij}}\}$, and $\{m_{0_{ij}}\}$ are column vectors with maximum number of elements up to $\frac{1}{2}n(n+1)$ elements. These column vectors include all the diagonal and right upper off-diagonal elements of the matrices $M$, $M_A$, $K$, $K_A$, and $M_0$, respectively. Also, matrices $[U]$, $[w]$, $[G_1]$, $[G_2]$, $[G_3]$, and $[G_4]$ are rectangular matrices with the dimension $\frac{1}{2}n(n+1)$ row by $nm$ column and can be expressed in Eqs. (23-28); $\{\lambda_{pq}\}$ is a column vector with $nm$ elements:

$$[U]\{\lambda_{pq}\} = \left[\sum_{i=j=1}^{n}a_{ip}\sum_{t=1}^{n}a_{ti}\Phi_{tq} + \sum_{j=i+1}^{n}\sum_{i=1}^{n-1}a_{ip}\sum_{t=1}^{n}a_{tj}\Phi_{tq}\right]\{\lambda_{pq}\} \tag{23}$$

$$[w]\{\lambda_{pq}\} = \left[\sum_{i=j=1}^{n}a_{pi}\sum_{t=1}^{n}a_{it}\Phi_{tq} + \sum_{j=i+1}^{n}\sum_{i=1}^{n-1}a_{pj}\sum_{t=1}^{n}a_{it}\Phi_{tq}\right]\{\lambda_{pq}\} \tag{24}$$

$$[G_1]\{\lambda_{pq}\} = \left[\sum_{i=j=1}^{n}b_{ip}y_{qi} + \sum_{j=i+1}^{n}\sum_{i=1}^{n-1}b_{ip}y_{qj}\right]\{\lambda_{pq}\} \tag{25}$$

$$[G_2]\{\lambda_{pq}\} = \left[\sum_{i=j=1}^{n}b_{pi}y_{qi} + \sum_{j=i+1}^{n}\sum_{i=1}^{n-1}b_{pj}y_{qi}\right]\{\lambda_{pq}\} \tag{26}$$

$$[G_3]\{\lambda_{pq}\} = \left[\sum_{i=j=1}^{n}z_{ip}y_{qi} + \sum_{j=i+1}^{n}\sum_{i=1}^{n-1}z_{ip}y_{qj}\right]\{\lambda_{pq}\} \tag{27}$$

$$[G_4]\{\lambda_{pq}\} = \left[\sum_{i=j=1}^{n}z_{pi}y_{qi} + \sum_{j=i+1}^{n}\sum_{i=1}^{n-1}z_{pj}y_{qi}\right]\{\lambda_{pq}\} \tag{28}$$

where

$$p = 1 \text{ to } n \qquad q = 1 \text{ to } m$$

From Eq. (2), the dynamic constraint equation expressed in element form gives

$$[D]\{k_{ij}\} = [E]\{m_{ij}\} \tag{29}$$

where $[D]$ and $[E]$ are rectangular matrices with maximum number of elements up to $nm$ row by $\frac{1}{2}n(n+1)$ column. Both matrices $[D]$ and $[E]$ can be obtained from Eqs. (30) and (31)

$$[D]\{k_{ij}\} = \sum_{i=j=1}^{n}\Phi_{iq}\{k_{ii}\} + \sum_{j=i+1}^{n}\Phi_{jq}\{k_{ij}\} + \sum_{j=1}^{i-1}\Phi_{jq}\{k_{ji}\} \tag{30}$$

$$[E]\{m_{ij}\} = \sum_{i=j=1}^{n}\left(\sum_{s=1}^{m}\Phi_{is}\omega_{sq}^2\right)\{m_{ii}\} + \sum_{j=i+1}^{n}\left(\sum_{s=1}^{m}\Phi_{js}\omega_{sq}^2\right)\{m_{ij}\} + \sum_{j=1}^{i-1}\left(\sum_{s=1}^{m}\Phi_{js}\omega_{sq}^2\right)\{m_{ji}\} \tag{31}$$

where

$$i = 1 \text{ to } n \qquad q = 1 \text{ to } m$$

Substituting Eqs. (21) and (22) into Eq. (29) yields

$$[D]\{k_{A_{ij}}\} - [D]([U] + [w])\{\lambda_{pq}\} = [E]\left(\{m_{A_{ij}}\} - \{m_{0_{ij}}\}\right) + [E]([G_1] - [G_3])\{\lambda_{pq}\} \tag{32}$$

Furthermore, rearranging Eq. (32) gives

$$[H]\{\lambda_{pq}\} = [D]\{k_{A_{ij}}\} - [E](\{m_{A_{ij}}\} - \{m_{0_{ij}}\}) \tag{33}$$

where

$$[H] = [E]([G_1] - [G_3]) + [D]([U] + [w]) \tag{34}$$

Since $[H]$ is an $nm$ row by $nm$ column nonsingular matrix, the inverse matrix $[H^{-1}]$ exists. From Eq. (33), the Lagrange multiplier $\{\lambda_{pq}\}$ can be expressed in Eq. (35) as

$$\{\lambda_{pq}\} = [H^{-1}]\left[[D]\{k_{A_{ij}}\} - [E]\Big(\{m_{A_{ij}}\} - \{m_{0_{ij}}\}\Big)\right] \tag{35}$$

Substituting Eq. (35) into Eqs. (21) and (22) yields the desired mass and stiffness matrices

$$\{k_{ij}\} = \{k_{A_{ij}}\} - \Big([U]+[w]\Big)[H^{-1}]\left[[D]\{k_{A_{ij}}\} - [E]\Big(\{m_{A_{ij}}\} - \{m_{0_{ij}}\}\Big)\right] \tag{36}$$

$$\{m_{ij}\} = \{m_{A_{ij}}\} - \{m_{0_{ij}}\} + \Big([G_1]+[G_2]-[G_3]-[G_4]\Big) \times [H^{-1}]\left[[D]\{k_{A_{ij}}\} - [E]\Big(\{m_{A_{ij}}\} - \{m_{0_{ij}}\}\Big)\right] \tag{37}$$

Therefore, from Eqs. (36) and (37), the corrected mass and stiffness matrices, satisfying both the orthogonality requirement and the eigenvalue equation, are obtained using the element modification method. The interaction terms between the mass and stiffness matrices can be directly determined from Eqs. (36) and (37).

## Conclusions

An analytical dynamic model modification has been achieved using the element correction method. This method corrects both the mass and stiffness matrices simultaneously while enforcing the orthogonality and eigenvalue equation constraints. The interaction effects between the mass and stiffness matrices that are given in the final equations can be very important to the analytical dynamic engineer. Also, no iteration is required in order to ensure that the desired mass and stiffness matrices satisfy the dynamic constraint. This method is very useful to improve the analytical model based on an incomplete set of modal test data.

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# Application of Load-Dependent Vectors Bases for Dynamic Substructure Analysis

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## I. Introduction

THE classical approach to dynamic substructuring has been used to describe the internal motions of each substructure by a linear combination of substructure modes, with the implicit assumption that these modes satisfy a certain substructure eigenvalue problem. Three basic variants of the method have been developed depending on whether the modes of each substructure are obtained with its interface held fixed, free, or loaded. Since it is not possible to define a unique eigenvalue problem for a given substructure and because none of these methods can yield exact results for the actual structure using a truncated set of exact mode shapes, the advisability of using exact substructure mode shapes can be seriously questioned. The suggestions for improvements should therefore be directed toward two basic questions: how to select a set of substructure modes, and how to enforce geometric compatibility at substructure boundaries.

A new method of dynamic analysis for structural systems subjected to fixed spatial distribution of the dynamic load was recently introduced by Wilson et al.[1] as an economic alternative to the classical mode-superposition technique. The method of solution is based on a transformation to a reduced system of generalized Ritz coordinates using load-dependent transformation vectors. By using the superposition of load-dependent vectors, static correction components similar to those of the classical mode-acceleration method are directly computed. New computational variants used to generate load-dependent vectors have been presented recently.[2] The method has also been applied in the context of dynamic substructuring as a direct extension of the fixed interface synthesis of component modes, replacing substructure eigenvectors by load-dependent Ritz transformation vectors or Lanczos vectors.[3]

The purpose of this Note is to discuss practical implementation aspects of this dynamic substructuring method related to the algorithm used to generate transformation vectors and the corresponding evaluation of error norms. A new computational variant that combines the subspace generation of load-dependent vectors and the static condensation technique is developed to compute global modes which are not a summation of local modes or a combination of substructure modes.

Received April 11, 1988; revision received May 1, 1989. 

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